

# WHAT IS THE REAL MEANING OF THE FROISSART THEOREM? \*

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The Froissart bounds for amplitudes and cross sections are explained and reconsidered to clarify the role of different assumptions. It is the physical conditions of unitarity and of no massless exchanges, together with mathematical properties of the Legendre functions, that imply much softer high-energy asymptotics for elastic amplitudes at physical angles than for the same amplitudes at nonphysical angles. The canonical log-squared boundary for  $\sigma_{tot}$  appears only under the additional hypothesis that the amplitude at any nonphysical angle cannot grow faster than some power of energy. The Froissart results are further shown to admit some reinforcement. Comparison of the familiar and new Froissart-like restrictions with the existing data on  $\sigma_{tot}$  and diffraction slope at all available energies (including LHC) does not allow yet to unambiguously determine the asymptotic behavior of  $\sigma_{tot}$ , but shows that its current increase cannot be saturated (*i.e.*, maximally rapid).

## 1 What is the Froissart theorem?

The Froissart theorem (or Froissart bound) is known since 1961, after publication of the paper [1]. Its main statement says that the total cross section of two-hadron interaction cannot grow with energy faster than  $\log^2 E$ . Moreover, according to the common opinion frequently repeated in the literature, violation of this limit would mean violation of unitarity.

All preLHC experimental data on total cross sections look consistent with such expectation (see *e.g.*, Ref. [2]). This is especially impressive for the nucleon–(anti)nucleon scattering, where Tevatron and cosmic rays provided much higher energies than for any

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other hadron pairs (note, however, large uncertainties of the cosmic ray data). Nevertheless, the same data admit also “heretic” descriptions, with faster energy growth (*e.g.*, fit [3] has a power of energy in asymptotics).

It is reasonable, therefore, to reconsider the theoretical background of the Froissart theorem to answer the following questions:

- Why, at all, quantum theory may provide any bounds (the Froissart bounds (F.b.) in particular) for energy increase of amplitudes and cross sections?
- How does the specific bound  $\sim \log^2 E$  arise?

## 2 Main steps of Froissart’s construction

Let us briefly recall the derivation of the Froissart theorem. In his paper [1], Froissart used the following assumptions: a) unitarity condition; b) strong interactions as an object for discussion; c) elastic amplitudes satisfy the Mandelstam (double-spectral) representation or, at least, the single dispersion relation in the momentum transfer; d) total cross sections grow with energy.

From these assumptions Froissart deduced several bounds: a) for forward (or backward) amplitudes ( $< s \log^2 s$ , where  $s$  is the c.m. energy squared); b) for total cross sections, as a consequence of the optical theorem ( $< \log^2 s$ ); c) for fixed-angle amplitudes ( $< s^{3/4} \log^{3/2} / \sin^{1/2} \theta$ ; for simplicity, constant factors are omitted in all inequalities).

Derivation of those bounds begins with the familiar partial-wave decomposition

$$A(s, t) = \sum_{l=0}^{\infty} (2l+1) a_l(s) P_l(z), \quad (1)$$

where  $t = 2k^2(-1 + z)$ ,  $z = \cos \theta$ . Due to unitarity,  $|a_l(s)| < 1$ . One more inequality is generated by the dispersion relation. It provides the Gribov–Froissart representation for  $a_l$  [1, 4], due to which  $|a_l| < B(s) \exp(-l \alpha_0)$  at large  $l$ . The value of  $\alpha_0$  is related to the nearest  $t$ -channel singularity  $t_0 = 2k^2(-1 + \cosh \alpha_0)$ ; at high energy  $\alpha_0^2 \sim s^{-1}$ . We have thus two boundaries for  $|a_l|$ , which intersect at  $l = L$ . For  $\sigma_{tot}$  to grow,  $L$  should grow as well. It appears then that only sum with  $l \leq L$  is important at high energies, and the asymptotic behavior of the amplitude is completely determined by the behavior of  $L$ . For the forward/backward scattering amplitude, where  $|P_l(\cos \theta)| = |P_l(\pm 1)| = 1$ ,

$$|A(s, 0)| < \sum_{l=0}^L (2l+1) \sim L^2, \quad (2)$$

while for the fixed-angle amplitude, with  $|P_l(\cos \theta)| < \sqrt{2/(\pi l \sin \theta)}$ ,

$$|A(s, z)| < \sum_{l=0}^L (2l+1)/(l \sin \theta)^{1/2} \sim L^{3/2}/(\sin \theta)^{1/2}. \quad (3)$$

The value of  $L$  is determined by the relation  $1 = B(s) \exp(-L \alpha_0)$ . Dispersion relation implies  $B(s) \sim (s/s_0)^N$  with fixed  $s_0$ , and  $L \sim s^{1/2} \log(s/s_0)$ . This directly leads to Froissart's results.

### 3 Modified approach to Froissart's bounds

Froissart's approach was recently reconsidered and modified [5]. The new approach uses the following assumptions: a) unitarity condition, just as before; b) absence of massless particles (it is just this point that marks strong interactions, no other assumptions about properties of interaction are used); c) no assumptions on dispersion relations. This set of assumptions provide bounds for elastic amplitudes and total cross sections, which generalize Froissart's ones.

#### 3.1 Main steps of the modified approach

As before, we have the unitarity bound  $|a_l| < 1$ . To obtain another one, we can transform the usual relations

$$a_l(s) = \frac{\pi k}{2\sqrt{s}} \int_{-1}^{+1} A(s, \cos \theta') P_l(\cos \theta') d(\cos \theta') = -\frac{ik}{2\sqrt{s}} \oint A(s, z') Q_l(z') dz', \quad (4)$$

where  $Q_l$  is the Legendre function of the 2nd kind. The closed integration contour runs anticlockwise around the cut of  $Q_l(z')$  between  $-1$  and  $+1$  (it is the only cut of  $Q_l(z')$  at positive integer  $l$ ). We can choose the contour to be an ellipse with  $z' \equiv \cosh(\alpha + i\phi')$ ,  $\alpha = \text{const} > 0$ ,  $-i dz' = \sin(\alpha + i\phi') d\phi'$ . The contour may be blown up, until it touches the nearest singularity with  $\alpha = \alpha_0 > 0$  (in the physical region  $\alpha = 0$ ). Then, at large  $l$ , we can apply the inequality (equivalent to one used by Froissart)

$$|\sin(\alpha + i\phi') Q_l(z')| < e^{-\alpha(l+1/2)} \sqrt{\frac{\pi}{2} \cosh \alpha} \quad (5)$$

(the right-hand side independent of  $\phi'$ !), to obtain  $|a_l| < B_0(s) \exp(-l\alpha_0)$ , where  $B_0(s)$  is determined by the nonphysical amplitudes  $A(s, z')$  on the integration contour. Again,

we obtain “critical” value  $L$ , satisfying the relation

$$e^{\alpha_0 L} = \frac{1}{2} B_0(s), \quad (6)$$

and the whole set of Froissart’s bounds in terms of  $L$ . Specific form of  $s$ -asymptotics depends on asymptotics of  $B_0(s)$ . Original Froissart’s results [1] are reproduced if  $B_0(s)$  grows as a power of  $s$ .

Though  $B_0(s)$  and  $B(s)$  look differently, they both are determined by the amplitudes  $A(s, t)$  with nonphysical values of  $t$  and should, thus, have similar high- $s$  behavior. The power dependence of  $B(s)$  was motivated by dispersion relations which, however, have never been generally proved. The modified approach does not need any dispersion relation, and asymptotics of  $B_0(s)$  becomes a completely independent assumption, not related neither to unitarity nor to analyticity. Moreover, as explained in Ref. [5], phenomenological linearity of hadronic Regge trajectories gives an indirect evidence for  $B_0(s)$  increasing faster than any power of  $s$ , *i.e.*, for  $\sigma_{tot}$  increasing faster than  $\log^2 s$ .

In any case, high-energy asymptotics for physical amplitudes (and cross sections) is much softer than that for nonphysical amplitudes. It is a very general consequence of the assumptions described above.

### 3.2 Enhancing the Froissart bounds

The inequality (5) is rather loose. Its more exact and stricter form is

$$|\sin(\alpha + i\phi') Q_l(z')| < e^{-\alpha(l+1/2)} \sqrt{\frac{\pi}{2l}} \cosh \alpha. \quad (7)$$

This leads to a different value of  $L$ , as determined by the relation

$$e^{\alpha_0 L} \sqrt{L} = \frac{1}{2} B_0(s). \quad (8)$$

The new value is smaller than the previous one, from Eq.(6), and leads to stricter boundaries for physical amplitudes and cross sections. For example, if  $B_0(s) \sim s^N$  (as usually assumed), then  $\sigma_{tot}$  cannot increase as  $\sim \log^2(s/s_0)$  with a fixed scale  $s_0$  (as usually stated). Instead, the scale  $s_0$  should itself increase with energy (as some power of  $\log s$ ).

### 3.3 New Froissart-like inequalities

Originally, Froissart [1] obtained restrictions for the forward/backward amplitudes ( $< L^2$ ) and for the fixed-angle ones ( $< L^{3/2}/\sin^{1/2} \theta$ ). Analysis of Ref. [5] gave also inequalities

for fixed- $t$  cases, not considered by Froissart. In particular, for physical (negative) values of  $t$  it gives  $|A(s, t)| < L^{3/2}(s/|t|)^{1/4}$  (for all inequalities here we omit constant factors).

Of course, all those inequalities provide restrictions, not prescriptions. If, nevertheless, the forward amplitude (as well as  $\sigma_{tot}$ ) is saturated, *i.e.*, grows with energy as fast as possible, then it definitely grows faster than any physical fixed- $t$  amplitude. This necessitates existence of the shrinking diffraction peak. The slope  $b$  of this peak should grow just as  $\sigma_{tot}$  or even faster [5].

Therefore, at high energies the ratio  $\sigma_{tot}/b$  cannot increase, it should either decrease or be a constant. It is worth to emphasize that this conclusion is very general: it is the result of the above mentioned assumptions of unitarity and absence of massless particles (sure, both are true for the strong interactions), appended by the not evident assumption of saturated total cross section.

This result has an interesting physical meaning. If one considers the hadron-hadron high-energy scattering as diffraction on a screen, then the slope  $b$  is proportional to the screen area, while the ratio  $\sigma_{tot}/b$  is proportional to the average blackness of the screen. Thus, in the high-energy asymptotics the average blackness should be either constant or decreasing. The former case might, in particular, correspond to completely black hadrons (as usually assumed).

The latter case means that the screen area increases faster than  $\sigma_{tot}$  and the screen, in average, becomes more transparent. This seems paradoxical, but for comparison we can recall the case of a constant  $\sigma_{tot}$ , where a constant slope in strong interaction scattering would contradict analyticity and  $t$ -channel unitarity [6]. The contradiction may be overcome by the Reggeon description (with the unit intercept), in which the target has an infinitely increasing radius (and area) and becomes more and more transparent. This case corresponds to  $\sigma_{tot}/b \sim 1/\log s$  and shows that a hadron at high energies may tend to become completely transparent, contrary to intuitive expectations for strong interactions.

#### 4 Current experimental situation

The LHC measurements of total and elastic cross sections [7] are a great progress. Their results agree quite well with the log-squared behavior of  $\sigma_{tot}$ , but they do not provide yet an unambiguous answer and still may be described “heretically”, with power increase [8].

More definite (and intriguing) is the energy behavior of the ratio  $\sigma_{tot}/b$  shown in Fig.1. At energies from 10 GeV to 100 GeV this ratio is nearly constant (or slightly

decreasing). However, it definitely increases when going to the LHC energy.

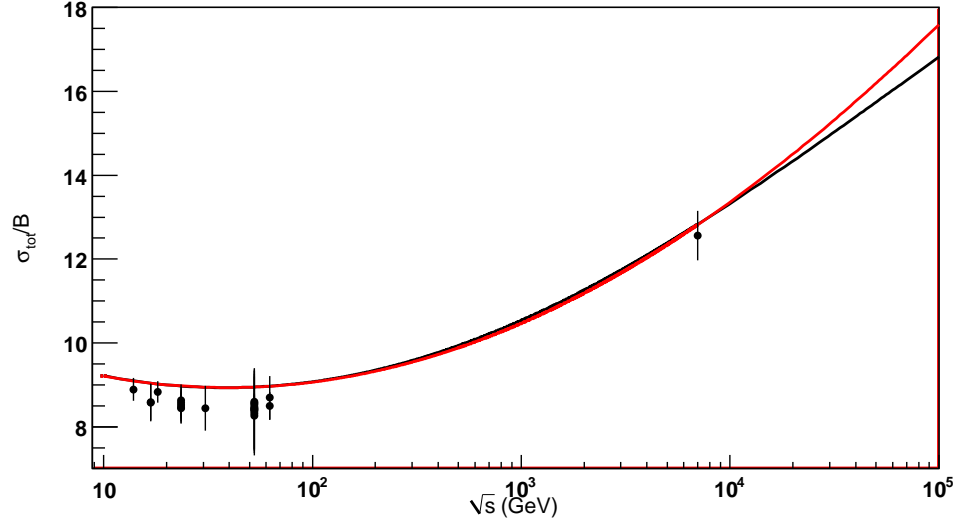


Figure 1: The energy dependence of the ratio  $\sigma_{tot}/b$  (the figure is taken from Ref. [9]).

According to the above analysis, such a result, if confirmed, means that the observed growth of  $\sigma_{tot}$  cannot be saturated.

## 5 Conclusions

Let us summarize results of the analysis [5].

- The very general result, which is *the real meaning* of the Froissart theorem, is the much softer energy growth for physical amplitudes *vs.* nonphysical ones. This is based on the physical assumptions of unitarity and absence of massless particles together with mathematical properties of the Legendre functions.
- Particular form of high-energy asymptotics of  $\sigma_{tot}$  is, theoretically, an open question. Commonly believed log-squared one is related to an additional suggestion (*never justified*) of not-stronger-than-power growth for amplitude(s) in any non-physical configurations. Violation of the log-squared behavior would not violate unitarity; it would contradict only to the additional asymptotic assumption.

- Familiar Froissart bounds may be enhanced; under the familiar assumptions,  $\sigma_{tot}$  can not grow faster than  $\ln^2(s/s_0)$  with a growing scale  $s_0$ , *i.e.*, must grow *slower* than canonically assumed.
- There are indirect arguments for cross sections to grow with energy *faster* than the “canonical” log-squared one.
- The observed relation between  $\sigma_{tot}$  and the diffraction slope  $b$  gives evidence that the presently observed energy growth is *not saturated*.
- Further studies, both theoretical and experimental, are necessary.

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